Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2015-2016
Semester II : Backpaper Examination
Probability III (Stochastic Processes)

15.07.2016 Time: 3 hours. Maximum Marks : 100

Note: Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. (20 marks) Let $\{X_n : n \geq 0\}$ be a time-homogeneous Markov chain on a countable state space S. Let $y \in S$ be fixed. Put $T_y^{(1)} = T_y = \min\{n \geq 1 : X_n = y\}$, $T_y^{(2)} = \min\{n > T_y^{(1)} : X_n = y\}$. For any $x \in S, m, n \geq 1$ show that
$$P_x(T_y^{(1)} = m, T_y^{(2)} = m + n) = P_x(T_y = m) \cdot P_y(T_y = n).$$

2. (20 marks) Let $M > 0$ be a fixed integer. Let $\{X_n : n \geq 0\}$ be a Markov chain on $\{0, 1, \cdots, M\}$ such that (i) $0, M$ are absorbing states, and (ii) $P_{i,i+1} = p_i > 0, P_{i,0} = (1 - p_i) > 0,\ 1 \leq i \leq (M - 1)$. For $1 \leq k \leq (M - 1)$ find the probability of absorption at 0 starting from k.

3. (15 + 10 = 25 marks) (i) Let $\{X_n\}$ denote the Ehrenfest urn model with a total of $2d$ balls. Show that the binomial distribution with parameters $2d$ and $\frac{1}{2}$ is the unique stationary probability distribution.

   (ii) Show that the Markov chain in (i) is time-reversible.

4. (15 marks) Let $\{N(t) : t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(\cdot)$. Put $m(t) = \int_0^t \lambda(r)dr,\ t \geq 0$. Assume that $\lambda(t) > 0$ for all t, and that $m(t) \to \infty$ as $t \to \infty$. Define $N^*(t) = N(m^{-1}(t)), t \geq 0$ where $m^{-1}(\cdot)$ is the inverse function of $m(\cdot)$. Show that N^* is a well-defined time homogeneous Poisson process.

5. (10 + 10 = 20 marks) Let $\{N(t) : t \geq 0\}$ be a time-homogeneous Poisson process with rate $\lambda > 0$. For $n = 1, 2, \cdots$ let W_n be the waiting time until the n-th event.

   (i) Show that $P(W_n < \infty) = 1$ for any fixed $n \geq 1$.

   (ii) Let $0 \leq s < t$ and $n \geq 1$. Find $P(W_1 < s | N(t) = n)$.